The Four-Vertex Theorem and The Decomposition of Polygons

Wiktor J. Mogilski

University of Texas at Brownsville

Discrete Geometry Workshop (2009)
Some Historical Background

The Smooth Four-Vertex Theorem:
S. Mukhopadhayaya (1909)
Adolf Kneser (1912)
R. Osserman (1985)

The Discrete Four-Vertex Theorem:
A. L. Cauchy (1813)
A. D. Aleksandrov (1950)
S. Bilinski (1963)
B. Dahlberg (2008)
R. C. Bose (1932)
Igor Pak (2008)

...
A Taste Of Things To Come

- Defining Three Types of Extremality
- Three Discrete Four-Vertex Theorems
- The Decomposition of Polygons and Some New Results
- Deriving the Four-Vertex Theorems
We will denote by $P$ a polygonal curve, which is just simply a piecewise linear curve, with vertices $V_1, V_2, \ldots, V_n$, $C_i = C(V_{i-1}V_iV_{i+1})$ the circumcircle formed by the corresponding vertices, and $R_i$ the radius of $C_i$. When we speak of a closed polygonal curve, we will refer to it as a polygon. All indices will be taken modulo the number of vertices of the polygonal curve.
We say that a polygonal curve is generic if the maximal number of vertices that lie on a circle is three and no three vertices are collinear.

Observe that every regular polygon is not generic.

A polygonal curve $P$ is coherent if for any three consecutive vertices $V_{i-1}$, $V_i$, and $V_{i+1}$, the center of the circle $C_i$ lies in the infinite cone formed by $V_{i-1}$, $V_i$, and $V_{i+1}$.

Note that all convex right and obtuse polygons are coherent, as well as all regular polygons.
Coherent vs. Non-Coherent
**Definition**

Let $C_{ijk}$ be a circle passing through any three vertices $V_i, V_j, V_k$ of a polygonal curve. We say that $C_{ijk}$ is empty if it contains no other vertices of the polygonal curve in its interior, and we say that it is full if it contains all of the other vertices of the polygonal curve in its interior.
Definition (Global Extremality)

Let $C_i$ be the circle passing through vertices $V_{i-1}$, $V_i$ and $V_{i+1}$. If $C_i$ is full or empty, then we call $C_i$ an extremal circle. We refer to the corresponding vertex $V_i$ as a globally extremal vertex. If $C_i$ is full, then we say that $V_i$ is a globally minimal-extremal vertex and if $C_i$ is empty, then we say that $V_i$ is a globally maximal-extremal vertex.

Remark.

Let $P$ be a polygonal curve. We denote the number of globally maximal-extremal vertices of $P$ by $s_-(P)$ and globally minimal-extremal vertices by $s_+(P)$. 

Wiktor J. Mogilski (University of Texas at Brownsville)
Definition (Delaunay Edge)

We call an edge or diagonal of a polygon a Delaunay edge if there exists an empty circle passing through the corresponding vertices of that edge or diagonal.

Definition (Anti-Delaunay Edge)

We call an edge or diagonal of a polygon an Anti-Delaunay edge if there exists a full circle passing through the corresponding vertices of that edge or diagonal.
A vertex $V_i$ is said to be positive if the left angle with respect to orientation $\angle V_{i-1}V_iV_{i+1}$ is at most $\pi$. Otherwise, it is said to be negative.
Assume that a vertex $V_i$ is positive. We say that the curvature of the vertex $V_i$ is greater than the curvature at $V_{i+1}$ ($V_i \succ V_{i+1}$) if the vertex $V_{i+1}$ is positive and $V_{i+2}$ lies outside the circle $C_i$ or if the vertex $V_{i+1}$ is negative and $V_{i+2}$ lies inside the circle $C_i$. 
By switching the word “inside” with the word “outside” in the above definition (and vice-versa), we obtain that $V_i \prec V_{i+1}$, or that the curvature at $V_i$ is less than the curvature at $V_{i+1}$.
In the case that the vertex $V_i$ is negative, simply switch the word “greater” with the word “less”, and the word “outside” by the word “inside”. The following figure shows a case where $V_i \prec V_{i+1}$:
Local Extremality

Definition (Local Extremality)

A vertex $V_i$ of a polygonal curve $P$ is locally extremal if

$$V_{i-1} \prec V_i \succ V_{i+1} \text{ or } V_{i-1} \succ V_i \prec V_{i+1}.$$ 

We say that $V_i$ is locally maximal-extremal if $V_{i-1} \prec V_i \succ V_{i+1}$ and locally minimal-extremal if $V_{i-1} \succ V_i \prec V_{i+1}$. We denote the number of locally maximal-extremal vertices of $P$ by $l_-(P)$ and locally minimal vertices by $l_+(P)$. 

Weigtor J. Mogilski (University of Texas at Brownsville)
The Four-Vertex Theorem and The Decomposition
April 18, 2009 15 / 40
Observing the definition of locally extremal vertices closely, we can see that if we assume convexity on our polygon, we really are considering the position of the vertices $V_{i-2}$ and $V_{i+2}$ with respect to the circle $C_i$. Our vertex $V_i$ will be locally extremal if they both lie inside or both lie outside the circle $C_i$. 
Let $P$ be a polygonal curve. Denote by $R_{i-1}$, $R_i$, and $R_{i+1}$ the radii of the circles $C_{i-1}$, $C_i$, and $C_{i+1}$, respectively.

**Definition (Radial Extremality)**

We say that a vertex $V_i$ is radially extremal if

$$R_{i-1} < R_i > R_{i+1}$$

or

$$R_{i-1} > R_i < R_{i+1}.$$

We say that a vertex $V_i$ is radially minimal-extremal if $R_{i-1} > R_i < R_{i+1}$ and radially maximal-extremal if $R_{i-1} < R_i > R_{i+1}$.
Proposition

Let $P$ be a generic convex polygon. If $V_i$ is a globally extremal vertex, then $V_i$ is a locally extremal vertex.
Local Extremality Does Not Imply Global Extremality
Proposition

Let $P$ be a generic convex coherent polygon. Then:

1. $V_{i-1} \succ V_i \iff R_{i-1} < R_i$
2. $V_{i-1} \prec V_i \iff R_{i-1} > R_i$

where $R_{i-1}$ is the radius of the circle $C_{i-1}$ and $R_i$ is the radius of the circle $C_i$.

Proof.

Follows directly from the Law of Sines.
The Four-Vertex Theorem In Three Flavors

Theorem (The Global Four-Vertex Theorem)

Every generic convex polygon with four or more vertices has at least four globally extremal vertices.

Theorem (The Local Four-Vertex Theorem)

Every generic convex polygon with four or more vertices has at least four locally extremal vertices.

Theorem (The Radial Four-Vertex Theorem)

Every generic coherent convex polygon with four or more vertices has at least four radially extremal vertices.
Important Facts

Proposition (Existence Proposition)

*Every generic convex polygon has at least one of each type of maximal-extremal vertex.*

Lemma (Quadrilateral Lemma)

*Let P be a generic convex quadrilateral. Then P has four globally extremal and locally extremal vertices.*

Lemma (Induction Lemma)

*Let P be a convex generic polygon with at least five vertices, and let \( V_i \) be a globally maximal-extremal vertex. Let \( P' \) be the polygon obtained by removing \( V_i \) and joining the vertices \( V_{i-1} \) and \( V_{i+1} \) by an edge. Then either \( s_-(P) = s_-(P') \) or \( s_-(P) = s_-(P') + 1. \)

Note that our Induction Lemma holds also for globally minimal-extremal vertices, as well as locally extremal vertices.
So what exactly does it mean to decompose a polygon? Here, the notion of decomposing a polygon will simply be the cutting of a polygon $P$ by passing a line segment through any two vertices so that the line segment lies in the interior of the polygon. We will call this line segment a \textit{diagonal}. Also, we will denote the two new polygons formed by a decomposition by $P_1$ and $P_2$ and also require that they each have at least four vertices.
Decomposition and Globally Extremal Vertices

Theorem (Global Inequality)

Let $P$ be a generic convex polygon with six or more vertices and let $P_1$ and $P_2$ be the resulting polygons of a decomposition. Then

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 3.$$
Why not $s_-(P) \geq s_-(P_1) + s_-(P_2) - 2$?

It is not too difficult to check that $A$ is maximal in $P'$ and not maximal in $P'_2$. Also, $D$ is not maximal in $P'$ and $P'_2$. So, we are actually in the “bad” case of our proof. After rigorously checking, we can verify that the globally extremal vertices of $P$ are $C$, $E$, and $G$, of $P_1$ are $L$ and $B$, and of $P_2$ are $C$, $G$, $I$ and $E$. So, $s_-(P) = 3 \geq s_-(P_1) + s_-(P_2) - 3 = 2 + 4 - 3 = 3$.
A Stronger Inequality

**Theorem (Stronger Global Inequality (Max))**

Let $P$ be a generic convex polygon with six or more vertices. Assume that the cutting diagonal of a decomposition is Delaunay. Then

$$s_-(P) \geq s_-(P_1) + s_-(P_2) - 2$$

**Theorem (Stronger Global Inequality (Min))**

Let $P$ be a generic convex polygon with six or more vertices. Assume that the cutting diagonal of a decomposition is Anti-Delaunay. Then

$$s_+(P) \geq s_+(P_1) + s_+(P_2) - 2$$

**Corollary (The Global Four-Vertex Theorem)**

Let $P$ be a generic convex polygon with six or more vertices. Then

$$s_+(P) + s_-(P) \geq 4.$$
Decomposition and Locally Extremal Vertices

Now that we have investigated what happens with globally extremal vertices, a natural question is, what happens with locally extremal vertices? Luckily, we will not have the situation as with globally extremal vertices. In fact, it is easy to see that the only vertices that will be affected by a decomposition of a polygon will be the vertices on the cutting diagonal and the neighboring vertices, a total of six vertices.

![Diagram of Decomposition of Polygon](image)
The Local Inequality

Theorem (Local Inequality)

Let \( P \) be a generic convex polygon with six or more vertices and let \( P_1 \) and \( P_2 \) be the resulting polygons of a decomposition. Then

\[
I_- (P) \geq I_- (P_1) + I_- (P_2) - 2.
\]

Corollary (The Local Four-Vertex Theorem)

Let \( P \) be a generic convex polygon with six or more vertices. Then

\[
I_+ (P) + I_- (P) \geq 4.
\]
Lemma (1)

Let $P$ be a generic convex polygon and $B$ and $D$ the vertices of a cutting diagonal. Let $A$ and $C$ be the neighbors of $B$ in $P$ and let $P_1$ and $P_2$ be the polygons obtained after a decomposition, with $P_1$ possessing vertex $A$ and $P_2$ possessing vertex $C$. Assume that $A$ is locally maximal-extremal in $P_1$ but not in $P$, and that $C$ is locally maximal-extremal for $P_2$ but not in $P$. Then, $B$ is a locally maximal-extremal vertex for $P$. 

![Diagram of Lemma (1)]
Lemma (2)

Let $P$ be a generic convex polygon and $B$ and $D$ the vertices of a cutting diagonal. Let $A$ and $C$ be the neighbors of $B$ in $P$ and let $P_1$ and $P_2$ be the polygons obtained after a decomposition, with $P_1$ possessing vertex $A$ and $P_2$ possessing vertex $C$. Assume that $A$ is locally maximal-extremal in $P_1$ but not in $P$, and that $B$ is locally maximal-extremal in $P_2$. Then, $B$ is locally maximal-extremal in $P$. 

![Diagram of a polygon with vertices A, B, C, and P1, P2]
Lemma (3)

Let \( P \) be a generic convex polygon and \( B \) and \( D \) the vertices of a cutting diagonal. Let \( A \) and \( C \) be the neighbors of \( B \) in \( P \) and let \( P_1 \) and \( P_2 \) be the polygons obtained after a decomposition, with \( P_1 \) possessing vertex \( A \) and \( P_2 \) possessing vertex \( C \). Assume that \( A \) is locally maximal-extremal for \( P_1 \) and \( D \) is locally maximal-extremal for both \( P_1 \) and \( P_2 \), but not for \( P \). Then \( A \) is locally maximal-extremal for \( P \).
Case 1.
We gain two maximal-extremal vertices in $P_1$, as well as $P_2$, but none of the six vertices are maximal-extremal in $P$.

Case 2.
We gain two maximal-extremal vertices in $P_1$ and gain two maximal-extremal vertex in $P_2$, and one of the six vertices is maximal-extremal in $P$.

Case 3.
We gain two maximal-extremal vertices in $P_1$ and gain one maximal-extremal vertex in $P_2$, and none of the six vertices is maximal-extremal in $P$. 
Lemma (Hexagon Lemma)

Let $P$ be a generic convex polygon with six vertices. Then $P$ has at least four globally extremal vertices, as well as at least four locally extremal vertices.

Proof.

Since $P$ has 6 vertices, if we were to decompose $P$ into $P_1$ and $P_2$, it would follow that $P_1$ and $P_2$ each have exactly four vertices. By our Quadrilateral Lemma, we have two maximally extremal vertices, and since they cannot be neighbors, from our global and local inequality our assertion follows.
Lemma (Triangulation Lemma)

Let $P$ be a convex polygon with seven or more vertices and let $T(P)$ be a triangulation of $P$. Then, there exists a diagonal of our triangulation such that if we apply a decomposition of $P$ using this diagonal, then both $P_1$ and $P_2$ have four or more vertices.

This lemma holds for any non-overlapping triangulation of $P$, in particular, a Delaunay triangulation.
Proof.

We perform induction on the number of vertices. Our Hexagon Lemma takes care of the case where $n = 6$. By our Triangulation Lemma, we know that there exists a diagonal such that, if we apply a decomposition by this diagonal, $P_1$ and $P_2$ will each have at least four vertices. Applying induction to $P_1$ and $P_2$ and using our Strong Global Inequality, we obtain our result.
Proof.

We perform induction on the number of vertices. Our Hexagon Lemma takes care of the case where $n = 6$. For the inductive step, we decompose our polygon $P$. Applying induction to $P_1$ and $P_2$ and using our Local Inequality, we obtain our result.


Igor Pak, Lectures on Discrete and Polyhedral Geometry, 183-197.