Linear Functionals

All material from Chapter 3 and 8 of Linear Algebra by Hoffman and Kunze.

**Definition:** Let $V$ be a vector space over a field $F$. Since $F$ is a vector space over $F$ we may consider the set $L(V, F)$. Elements of $L(V, F)$ are called **linear functionals.** $L(V, F)$ is called the dual space of $V$ and is denoted by $V^*$. We know that $V^*$ is a vector space over $F$ and that if $V$ is finite dimensional then $\dim(V) = \dim(V^*)$. This tells us that $V$ and $V^*$ are isomorphic if $\dim(V) < \infty$. Shortly we will find a nice isomorphism.

**Three important examples:** These are probably the three most important examples of where linear functionals come from.

1. Suppose that $F$ is a field, and $S$ is a non-empty set. We know that $V = \{f : S \rightarrow V\}$ is a vector space. For each $s \in S$ we may define a linear functional $L_s$ by
   $$L_s(f) = f(s).$$
   These are called the evaluation linear functionals.

2. Linear functionals on inner product spaces. Suppose that $V$ is an inner product space. Then each $\beta \in V$ defines a linear functional $L_\beta$ given by
   $$L_\beta(\alpha) = (\alpha|\beta).$$
   For example, if $F = C$, the complex numbers, $(\cdot|\cdot)$ is the standard inner product on $C^3$, $\beta = (2 + i, 3 - i, i)$, and $\alpha = (x, y, z)$ then
   $$L_\beta(\alpha) = ((x, y, z)|(2 + i, 3 - i, i)) = (2 - i)x + (3 + i)y - iz.$$  
   For another example, suppose that $V$ is the real valued continuous functions with domain $[-1, 1]$, the field is the real numbers, the inner product is the standard inner product, and $\beta$ is the constant function 1. Then $L_1$ is given by
   $$L_1(f) = \int_{-1}^{1} f(x) \, dx.$$  

3. Determinants. Suppose that $F$ is a field, $n > 1$ is an integer and $\beta_2, \ldots, \beta_n \in F^n$ are given to us. Then for each $\alpha \in F^n$ we may define a matrix whose first row is $\alpha$, second row is $\beta_2$ and so on. Then since the determinant function is $n$-linear, the function $L : F^n \rightarrow F$ defined by
   $$L(\alpha) = \det(\alpha, \beta_2, \ldots, \beta_n)$$
   is a linear functional. For example, if $n = 2$, $\beta_2 = (2, 5)$, and $\alpha = (x, y)$ then
   $$L(\alpha) = L((x, y)) = \det \begin{bmatrix} x & y \\ 2 & 5 \end{bmatrix} = 5x - 2y$$
   Another example. If $n = 3$, $\beta_2 = (3, 4, 7)$, $\beta_3 = (2, 1, 4)$ and $\alpha = (x, y, z)$ then
   $$L(\alpha) = L((x, y, z)) = \det \begin{bmatrix} x & y & z \\ 3 & 4 & 7 \\ 2 & 1 & 4 \end{bmatrix} = 9x + 2y - 5z,$$
   We shall see that this last example is related to cross product.

**Definition:** Suppose that $B$ is a basis for $V$. Then for each $\beta \in B$ there is a unique (see Theorem 3.1) linear functional $\beta^* \in V^*$ such that $\beta^*(\beta) = 1$ and $\beta^*(\alpha) = 0$ if $\alpha \in B$ and $\alpha \neq \beta$. $\beta^*$ is called the dual of $\beta$. 

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Lemma: If $B$ is a linearly independent set in $V$ then the set $\{\beta^* \in V^*\}$ is a linearly independent
subset of $V^*$.

Demonstration: Let $Z \in V^*$ be the zero element of $V^*$, that is $Z(\alpha) = 0$ for every $\alpha \in V$. Suppose that

$$c_1\beta^*_1 + \cdots + c_n\beta^*_n = Z.$$ 

Then

$$c_1\beta^*_1(\beta_1) + \cdots + c_n\beta^*_n(\beta_1) = Z(\beta_1)
\quad (c_1 \times 1) + (c_2 \times 0) \cdots + (c_n \times 0) = 0$$

so $c_1 = 0$. In the same way, $c_2 = \cdots = c_n = 0$.

Definition: If the collection $\{L_\beta\}$ is a basis of $V^*$ it is called the dual basis.

Theorem 3.15 Let $V$ be a finite dimensional vector space over the field $F$. Let $(\beta_1, \ldots, \beta_n)$ be
an ordered basis for $V$, and put $\beta_k = L_{\beta_k}$. Let $I$ denote the $n \times n$ identity matrix in $F^{n \times n}$. Then $(\beta_1, \ldots, \beta_n)$ is the unique ordered basis in of $V^*$ with the following properties:

1. $\beta_k^*(\beta_j) = I_{k,j}$ for $k, j \in \{1, \ldots, n\}$;
2. If $\beta^* \in V^*$ then

$$\beta^* = \sum_{k=1}^n \beta^*(\beta_k)\beta_k^*$$

3. If $\alpha \in V$ then

$$\alpha = \sum_{k=1}^n \beta^*(\alpha)\beta_k.$$

Corollary: Let $V$ is a finite dimensional vector space over $F$ with ordered basis

$$B = \{\beta_1, \ldots, \beta_n\}.$$ 

Let

$$B^* = \{\beta_1^*, \ldots, \beta_n^*\}$$

be the corresponding dual basis for $V^*$. Then for each $\alpha \in V$ and $f \in V^*$

$$[f]_{B^*} = \begin{bmatrix} f(\beta_1) \\
\vdots \\
f(\beta_n) \end{bmatrix}$$

$$[\alpha]_B = \begin{bmatrix} \beta_1^*(\alpha) \\
\vdots \\
\beta_n^*(\alpha) \end{bmatrix}.$$ 

$$[f] = \begin{bmatrix} f(\beta_1) & \cdots & f(\beta_n) \end{bmatrix}$$

$$f(\alpha) = \sum_{k=1}^n \beta_k^*(\alpha)f(\beta_k)$$

Riesz Representation Theorem: Let $S$ be a finite set of $n$ elements, let $F$ be a field, and let

$$V = \{f : S \to F\}.$$ 

Let $f \in V^*$. Then there is a unique $\phi \in V$ so that for all $\alpha \in F^n$

$$f(\alpha) = \sum_{s \in S} \alpha(s)\phi(s)$$

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**Corollary:** Let $V$ be a finite dimensional vector space over the field $F$ with ordered basis $(\beta_1, \ldots, \beta_n)$. Define $T: V \to V^*$ by

$$ T(\alpha) = \sum_{k=1}^{n} \beta_k^*(\alpha) \beta_k^* $$

and $S: V^* \to V$ by

$$ S(f) = \sum_{k=1}^{n} f(\beta_k) \beta_k. $$

Then $T$ is the isomorphism between $V$ and $V^*$ that carries each $\beta_k$ to its dual element $\beta_k^*$ and $S$ is its inverse.

**Interpolating Polynomials:** Let $F$ be a field, let $S \subset F$ be non-empty, and let $V = \{ f : s \to F \}$. Let $s_1, s_2, \ldots, s_n$ be $n \geq 1$ distinct elements in $S$ and define the polynomial functions

$$ p(x) = (x - s_1)(x - s_2) \cdots (x - s_n) $$

$$ p_1(x) = p(x)/(x - s_1) $$

$$ \vdots $$

$$ p_n(x) = p(x)/(x - s_n) $$

(We adopt the convention that the polynomial division on the right hand sides is done formally to avoid confusion about division by 0.) Form new polynomial functions $q_k$ by $q_k(x) = p_k(x)/p_k(s_k)$ so that $q_k(s_j) = I_{k,j}$. Then the polynomials $q_k$ are a basis for the space of polynomials of degree less than $n$ over $F$ and the evaluation functionals $L_k(f) = f(s_k)$ are their duals.

**Definition:** Suppose that $B = \{ \beta_1, \ldots, \beta_n \}$ is an ordered basis of $R^n$. We say that $B$ is **righthanded** if $\det(\beta_1, \ldots, \beta_n) > 0$. We say that $B$ is **lefthanded** if $\det(\beta_1, \ldots, \beta_n) < 0$.

**Cross Product:** Let $R$ denote the real numbers and let $V = R^3$. $V$ is the same as $\{ f : \{1, 2, 3\} \to R \}$. Let $\langle \cdot | \cdot \rangle$ be the standard inner product on $V$. Fix $\beta$ and $\gamma \in V$ and define $L \in V^*$ by $L(\alpha) = \det(\alpha, \beta, \gamma)$ as in the examples above. Then according to the Riesz Representation Theorem there is some $\phi \in V$ so that $L(\alpha) = \langle \alpha | \phi \rangle$. From the properties of determinants we know

$$ (\beta | \phi) = \det(\beta, \beta, \gamma) = 0 $$

$$ (\gamma | \phi) = \det(\gamma, \beta, \gamma) = 0 $$

Furthermore, $\phi = (0, 0, 0)$ if and only if $\beta$ and $\gamma$ are linearly independent, for

$$ 0 \leq (\phi | \phi) = \det(\phi, \beta, \gamma). $$

If $\beta$ and $\gamma$ are linearly dependent, then $\det(\phi, \beta, \gamma) = 0$, so $\phi = (0, 0, 0)$. If $\beta$ and $\gamma$ are linearly independent, then there is some $\alpha \in V$ so that $(\alpha, \beta, \gamma)$ are linearly independent, and, therefore, are the rows of an invertible matrix. Then

$$ (\alpha | \phi) = \det(\alpha, \beta, \gamma) \neq 0 $$

so $\phi \neq 0$ and

$$ \det(\phi, \beta, \gamma) = (\phi | \phi) > 0. $$

The vector $\phi$ is called the **cross product** of $\beta$ and $\gamma$ and is denoted by $\beta \times \gamma$.

If $\beta$ and $\gamma$ are linearly independent then $(\beta \times \gamma, \beta, \gamma)$ is a righthanded basis for $R^3$.

To see more properties of cross product, go to

http://www.uwm.edu/~ericskey/535material/crossprod.pdf
**Definition:** Suppose that $V$ is an inner product space and $f \in V^*$. We say that $f$ is **bounded** if there is a real number $M$ so that

$$|f(\alpha)| \leq M\|\alpha\|.$$

An inner product space $H$ is called a **Hilbert space** (named after the mathematician David Hilbert) if every for each bounded linear functional $f$ there is some $\phi \in H$ so that

$$f(\alpha) = (\alpha|\phi)$$

for every $\alpha \in H$. Note that phi must be unique for if $(\alpha|\phi) = (\alpha|\phi')$ for all $\alpha$ then $\phi = \phi'$.

**Theorem:** If $V$ is an inner product space, $f \in V^*$ and $f(\alpha) = (\alpha|\phi)$ for all $\alpha \in V$ then $V = N(f) \bigoplus N(f)^\perp$.

**Demonstration:** The theorem is clearly true if $f(\alpha) = 0$ for all $\alpha$. If not, then choose $\phi$ so that $f(\alpha) = (\alpha|\phi)$. We know that $\phi \neq 0$. If $\alpha \in V$ then

$$\alpha = \text{proj}_\phi(\alpha) + (\alpha - \text{proj}_\phi(\alpha)).$$

Since we have

$$f(\alpha - \text{proj}_\phi(\alpha)) = ((\alpha - \text{proj}_\phi(\alpha))|\phi)$$

we see that

$$\alpha - \text{proj}_\phi(\alpha) \in N(f)$$

$$\text{proj}_\phi(\alpha) \in N(f)^\perp$$

Since we know that for every subspace $W$, $W \cap W^\perp = \{\vec{0}\}$ we have proven our claim.

**Theorem 8.6:** Every finite dimensional inner product space is a Hilbert space.

**Demonstration:** Let $(\nu_1, \ldots, \nu_n)$ be an orthonormal basis of $V$ and put

$$\phi = \sum_{k=1}^n f(\nu_k)\nu_k$$

Then it is easy to check that $f(\alpha) = (\alpha|\phi)$ for every $\alpha \in H$.

**Definition:** If $V$ is a vector space, and $W$ and $W'$ are subspaces of $V$, we say that $W$ is a **hyperspace** or **hyperplane** if

- $V = W \bigoplus W'$;
- $\dim(W') = 1$.

For this reason a hyperspace is also called a **co-dimension one** since the dimension of the “complementary” space $W'$ is one. If the dimension of $V$ is $n$, then every subspace of dimension $n-1$ is a hyperspace. In an inner product space $V$, if $\alpha \neq 0$ and $V = \text{span} \{\alpha\}^\perp$ then $\{\alpha\}^\perp$ is a hyperspace. In a Hilbert space, the null space of every bounded linear functional that is not identically 0 is a hyperspace.

**Definition:** If $V$ is a vector space over a field $F$ and $S$ is a subspace of $V$, the **annihilator** of $S$ is the subspace of $V^*$, call it $S^0$, with the property that $f \in S^0$ if and only if $f(\sigma) = 0$ for each $\sigma \in S$. Observe that $\dim(V^0) = 0$ and $\{\vec{0}\}^0 = V^*$.

**Theorem 3.16:** If $V$ is a finite dimensional vector space and $W$ is a subspace of $V$ then

$$\dim(W) + \dim(W^0) = \dim(V).$$

**Corollary:** If $W$ is a $k$ dimensional subspace of an $n$ dimensional vector space $V$ then $W$ is the intersection of $(n-k)$ hyperspaces in $V$.

**Corollary:** If $W_1$ and $W_2$ are subspace of a finite dimensional vector space $V$ then $W_1 = W_2$ if and only if $W_1^0 = W_2^0$. 

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