Definition: A vector space consists of the following:

1. a field \( F \) of scalars;
2. a set \( V \) of objects called vectors;
3. An operation, called vector addition, which associates to each pair of vectors \( \alpha \) and \( \beta \) in \( V \) a vector \( \alpha + \beta \in V \), called the sum of \( \alpha \) and \( \beta \) in such a way that
   (a) vector addition is commutative;
   (b) vector addition is associative;
   (c) there is a vector \( \vec{0} \in V \), called the zero vector, such that \( \alpha + \vec{0} = \alpha \) for all \( \alpha \in V \).
   (d) for each \( \alpha \in V \) there is a vector \( -\alpha \in V \) so that \( -\alpha + \alpha = \vec{0} \).
4. an operation, called scalar multiplication, which associates with each scalar \( c \in F \) and each vector \( \alpha \in V \) a vector \( c\alpha \in V \), called the product of \( c \) and \( \alpha \), in such a way that
   (a) \( 1\alpha = \alpha \) for each \( \alpha \in V \);
   (b) \( (c_1c_2)\alpha = c_1(c_2\alpha) \);
   (c) \( c(\alpha + \beta) = c\alpha + c\beta \)
   (d) \( (c_1 + c_2)\alpha = c_1\alpha + c_2\alpha \).

Example: If \( F \) is a subfield of the field \( G \) then \( G \) is a vector space over \( F \), but \( F \) is not a vector space over \( G \) unless \( F = G \).

Example: If \( S \) is any non-empty set and \( W \) is a vector space over \( F \), then the set of all functions from \( S \) to \( W \) is a vector space over \( F \) with the usual definition of sum of functions and multiplication of functions by scalars. This vector space is sometimes denoted by \( W^S \). In particular, \( F^S \) is a vector space over \( F \).

Definition: A vector space \( V \) over \( F \) is called an inner product space if

1. \( F \) is a subfield of the complex numbers.
2. there is an operation, called the inner product, which associates to each pair of vectors \( \alpha \) and \( \beta \) a scalar \( \langle \alpha | \beta \rangle \in F \) so that
   (a) \( \langle c\alpha + \beta | \gamma \rangle = c\langle \alpha | \gamma \rangle + \langle \beta | \gamma \rangle \);
   (b) \( \langle \beta | \alpha \rangle \) is the complex conjugate of \( \langle \alpha | \beta \rangle \).
   (c) \( \langle \alpha | \alpha \rangle > 0 \) if \( \alpha \neq \vec{0} \).

Example 1: Suppose that \( V = F^{n \times 1} \) where \( F \) is a subfield of the complex numbers and \( w_k > 0 \) for \( k = 1, 2, \ldots, n \). Then
\[
\langle \alpha | \beta \rangle = \sum_{k=1}^{n} w_k \alpha_k \overline{\beta_k}
\]
defines an inner product on \( V \). The scalars \( w_k \) are called weights. If the weights are all 1 this is called the standard inner product. If \( F \) is contained in the real numbers and the weights are all 1 this is called the dot product.

Example 2: Suppose that \( V \) is the set of continuous functions from \([-1, 1]\) into the complex numbers, \( F \) is a subfield of the complex numbers, and \( w \in V \) and \( w(x) > 0 \) for \( x \in (-1, 1) \) then
\[
\langle \alpha | \beta \rangle = \int_{-1}^{1} \alpha(x) \overline{\beta(x)} w(x) \, dx
\]
defines an inner product on \( V \).
Example 3: Suppose that $T : V \to W$ is a one-to-one linear transformation and $W$ is an inner product space with inner product $(\cdot|\cdot)_W$. Then

$$(\alpha|\beta)_V = (T(\alpha)|T(\beta))_W$$

defines an inner product on $V$.

Proposition: Suppose that $V$ is a vector space over $F$.

1. $c\vec{0} = \vec{0}$;
2. $0\alpha = \vec{0}$;
3. $c\alpha = \vec{0}$ implies $c = 0$ or $\alpha = \vec{0}$.
4. $-1\alpha = -\alpha$

If $V$ is an inner product space then

1. $(\alpha|c\beta + \gamma) = \overline{c}(\alpha|\beta) + (\alpha|\gamma)$ where $\overline{c}$ denotes the complex conjugate of $c$.
2. $(\alpha|\alpha) = 0$ if and only if $\alpha = \vec{0}$.

Definition: If $V$ is an inner product space over $F$ then the norm of a vector $\alpha$, denoted by $\|\alpha\|$ is given by

$$\|\alpha\| = \sqrt{(\alpha|\alpha)}.$$ 

If $\|\alpha\| = 1$ we say that $\alpha$ is a unit vector. If $F$ contains the real numbers and $\alpha \neq \vec{0}$ then the vector $(1/\|\alpha\|)\alpha$ is called the direction of $\alpha$. The direction of $\alpha$ is a unit vector.

Polarization identities: Relations between the norm and the inner product.

1. If $F$ is a subfield of the real numbers then

$$4(\alpha|\beta) = (\alpha + \beta|\alpha + \beta) - (\alpha - \beta|\alpha - \beta)$$

so if $F$ is the real numbers then

$$(\alpha|\beta) = \frac{1}{4} \left(\|\alpha + \beta\|^2 - \|\alpha - \beta\|^2\right).$$

2. If $F$ is the complex numbers, then

$$(\alpha|\beta) = \frac{1}{4} \sum_{k=1}^{4} i^k \|\alpha + i^k\beta\|^2.$$ 

Definition $\alpha$ and $\beta$ are said to be orthogonal if $(\alpha|\beta) = 0$. A set of vectors is said to be orthogonal if any pair of vectors in the set is orthogonal. A set of orthogonal vectors is said to be orthonormal if each vector in the set is a unit vector.

Lemma: If $\alpha \neq \vec{0}$, define $\text{proj}_\alpha : V \to V$ by

$$\text{proj}_\alpha(\beta) := \frac{\langle \beta|\alpha \rangle}{\langle \alpha|\alpha \rangle}\alpha.$$ 

$\text{proj}_\alpha(\beta)$ is called the (orthogonal) projection of $\beta$ onto $\alpha$, and $\beta - \text{proj}_\alpha(\beta)$ is orthogonal to $\alpha$.

Pythagorean Theorem: If $\alpha$ and $\beta$ are orthogonal then

$$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha - \beta\|^2$$

Law of Cosines: If $\alpha \neq \vec{0}$ and $\beta \neq \vec{0}$ then

$$\|\alpha - \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 - 2\|\alpha\|\|\beta\|\frac{(\langle \alpha|\beta \rangle + \langle \beta|\alpha \rangle)/2}{\|\alpha\|\|\beta\|}.$$
Theorem 8.1: If $V$ is an inner product space then for any vectors $\alpha$ and $\beta$ and any scalar $c$:

**Norm scaling:** $\|c\alpha\| = |c| \cdot \|\alpha\|$.  
**Positivity:** If $\alpha \neq \vec{0}$ then $\|\alpha\| > 0$.  

**Cauchy-Schwarz-Bunyakowski Inequality:** $|\langle \alpha | \beta \rangle| \leq \|\alpha\| \cdot \|\beta\|$.  

**Triangle inequality:** $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$. 

The CSB Inequality is proven by applying the Pythagorean Theorem to $\text{proj}_\alpha(\beta)$ and $\text{proj}_\alpha(\beta) - \beta$.

**Definition:** Let $V$ be a vector space over the field $F$. A *subspace* of $V$ is a subset $W$ of $V$ which is itself a vector space over $F$ with respect to the operations of $V$.

**Theorem 2.1:** A non-empty subset $W$ of a vector space $V$ over $F$ is a subspace of $V$ if and only if $W$ is closed under scalar multiplication and vector addition, that is, for every pair of vectors $\alpha$ and $\beta$ in $W$ and each scalar $c \in F$ we have $c\alpha + \beta \in W$.

**Example:** Let $V$ be an inner product space, and let $\alpha \in V$ be given. $W = \{\beta \in V : \langle \beta | \alpha \rangle = 0\}$ is a subspace of $V$.

**Theorem 2.2:** The intersection of subspaces of $V$ is a subspace of $V$.

**Example:** If $W$ is a subset of an inner product space, the set $W^\perp := \{\beta \in V : \langle \beta | \omega \rangle = 0 \text{ for all } \omega \in W\}$ is a subspace called the *orthogonal complement* of $W$.

**Definition:** Let $S \subset V$. The subspace *spanned* by $S$ is the intersection of all subspaces of $V$ that contain $S$. If $S$ is empty or if $S = \{\vec{0}\}$ then the span of $S$ is $\{\vec{0}\}$.

**Definition:** If $S \subset V$, a vector $\beta$ is said to be a *linear combination* of the elements of $S$ if there are scalars $c_1, \ldots, c_n$ and vectors $\sigma_1, \ldots, \sigma_n$ in $S$ so that

$$\beta = \sum_{k=1}^{n} c_k \sigma_k$$

If $\emptyset \neq S \subset V$ we say $S$ is *linearly independent* if the only linear combination of distinct elements of $S$ that equals $\vec{0}$ is the one with all the scalars equal to 0.

**Observation:** If $S$ is an orthonormal set in $V$ and $\beta$ is a a linear combination of elements of $S$ then

$$\beta = \sum_{\sigma \in S} (\beta | \sigma) \sigma.$$

**Note:** Under the hypotheses, $(\beta | \sigma) \neq 0$ for only a finite number of elements of $S$.

**Theorem 8.2:** An orthonormal set is a linearly independent set.

**Theorem 2.3:** The subspace spanned by a non-empty set $S$ of a vector space $V$ is the set of all linear combinations of elements of $S$.

**Definition:** If $S_1, S_2, \ldots, S_k$ are subsets of a vector space $V$, the set of all sums $\sigma_1 + \cdots + \sigma_k$ where $\sigma_j \in S_j$ is called the *sum* of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by $S_1 + S_2 + \cdots + S_k$ or

$$\sum_{j=1}^{k} S_j.$$
Proposition: If $W_1, W_2, \ldots, W_n$ are subspaces of $V$ then $W_1 + W_2 + \cdots + W_n$ is the subspace spanned by $W_1 \cup W_2 \cdots \cup W_n$.

Definitions: If $A \in F^{m \times n}$ then the row vectors of $A$ are the elements of $F^m$ given by
\[ \rho_i = (A_{i,1}, \ldots, A_{i,n}) \]
while the column vectors of $A$ are the elements of $F^n$ given by
\[ \kappa_j = (A_{1,j}, \ldots, A_{m,j}) \]
The row space of $A$ is the subspace spanned by the row vectors of $A$, and the column space of $A$ is the subspace spanned by the column vectors of $A$. 