Limiting and Stationary Probabilities

We want to address the question of when $\Pr(X_n = k|X_0 = i)$ converges when $n \to \infty$.

1 Ergodic Markov Chains

There is no chance that $(P^n)_{i,i}$ converges to a positive limit if $i$ is not aperiodic. The same can be said if $i$ is transient. We need some additional conditions for interesting convergence results.

For a given recurrent state $i$, define $T_i = \min\{n : X_n = i\}$. If $E[T_i|X_0 = i] < \infty$ we say that the state $i$ is positive recurrent. If a state $i$ is both positive recurrent and aperiodic we say that the state is ergodic. It can be shown that the ergodic property is a class property (see Basic Probability Theory by Robert Ash).

We have the following theorem.

Theorem 1 (Ergodic Theorem) If the Markov chain $X_n$ with state space $S$ is irreducible and all its states are ergodic then

1. There exist unique probability mass function $\pi$ on $S$ such that for each $s \in S$
   \[
   \pi_s = \sum_{x \in S} \pi_x P_{x,s},
   \]
   or, in matrix form, $\vec{\pi} = \vec{\pi}P$.

2. For each $x \in S$
   \[
   \lim_{n \to \infty} (P^n)_{x,s} = \pi_s.
   \]
   In matrix form, there is a matrix $Q$ with all rows equal to $\vec{\pi}$ and with $P^n \to Q$ as $n \to \infty$.

3. For each $s \in S$ let $I_s(x) = 1$ if $x = s$ and 0 otherwise. Then
   \[
   \Pr \left( \lim_{n \to \infty} \frac{I_s(X_1) + \cdots + I_s(X_n)}{n} = \pi_s \right) = 1,
   \]
   that is, $\pi_s$ is the long-run average time the process spends in state $s$.

The hypotheses of this theorem are satisfied by all irreducible aperiodic finite state Markov chains.

One case where the $\pi_s$ are easy to calculate is when the transpose of the transition matrix is also a transition matrix. If the transition matrix is finite dimensional and the chain is irreducible then $\pi_s = 1/N$ where $N$ is the number of states. See, for example, problems 20, 21 and 22.

2 Stationary Distributions

We say that a probability mass function $\nu$ defined on the state space $S$ is stationary if it the case that if $\Pr(X_0 = s)\nu_s$ then $\Pr(X_1 = s) = \nu(s)$ for all $s$ in $S$. This is the same as saying the $\nu P = \nu$. Hence, any limiting distribution $\vec{\pi}$ is a stationary distribution, but not every stationary distribution is a limiting distribution. The ergodic theorem gives conditions where the limiting distribution is the only stationary distribution.
3 The gambler’s ruin problem

Consider the Markov chain on the the set \( \{0, 1, \ldots, N\} \) where \( N \) is at least 2 and with the transition matrix \( P_{0,0} = P_{N,N} = 1 \), and for \( s \neq 0, N \), \( P_{s,s-1} + P_{s,s+1} = 1 \). Let \( p = P_{s,s+1} \in (0,1) \). This Markov chain is called the gambler’s ruin for the following reason. Let \( X_n \) denote the number of dollars a gambler has. Each time he plays he wins a dollar with probability \( p \) and loses a dollar with probability \( 1 - p \). If \( X_n = 0 \) or \( X_n = N \) the game is over. The question is how to determine the probability that the state 0 is ever reached (it is absorbing) if the gambler starts at state \( s \).

Denote this probability by \( f_s \). It is clear that \( f_0 = 1 \) and \( f_N = 0 \). By considering one play of the game we see that \( f_s = (1 - p)f_{s-1} + pf_{s+1} \) for \( 0 < s < N \). To solve for \( f_s \), we guess a solution of the form \( f_s = r^s \). Then \( r^s = qr^{s-1} + pr^{s+1} \) where \( q = 1 - p \). Dividing through by \( r^{s-1} \) we see that \( r = q + pr^2 \), so \( r = 1 \) or \( r = q/p \). We consider first the case where \( q \neq p \). Then \( f_s = A + B(q/p)^s \). Since \( f_0 = 1 \) while \( f_N = 0 \) we can solve for \( A \) and \( B \) in terms of \( p, q \) and \( N \) to get

\[
 f_s = 1 - \frac{1 - (q/p)^s}{1 - (q/p)^N}. \tag{1}
\]

To see what happens when \( p = q \), let \( h = q/p \) and let \( h \to 1 \) in (1). Since

\[
 \frac{1 - r^s}{1 - r^N} = \frac{1 + r + \ldots + r^{s-1}}{1 + r + \ldots + r^{N-1}},
\]

we see that as \( h \) approaches 1, \( f_s \) approaches \((N - s)/N\) which we can show directly satisfies \( f_s = (1/2)f_{s-1} + (1/2)f_{s+1} \) as well as the boundary conditions \( f_0 = 1 \) and \( f_N = 0 \).