PART 1: CONSTRUCTION OF OUTER MEASURES

1. Show that any outer measure \( \mu_e \) on a set \( X \) is associated with a sequential cover \( Q \) of \( X \) and a function \( \lambda : Q \to [0, \infty] \). (Hint: take \( Q = 2^X \) and \( \lambda = \mu_e \).

**Proof.** We must show that for any set \( X \), any outer measure \( \mu_e \) on the subsets of \( X \), and any subset \( E \subseteq X \),

\[
\mu_e = \inf \left\{ \sum_{n=1}^{\infty} \mu_e(E_n) | E \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and } E_n \in 2^X \right\}
\]

Since \( 2^X \) is a \( \sigma \)-algebra, we can use the monotonicity and countable sub-additivity of \( \mu_e \) to write

\[
\mu_e(E) \leq \mu_e \left( \bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu_e(E_n),
\]

so the infimum on the right-hand side of (1) is not less than \( \mu_e(E) \). But the singleton \( \{E\} \) is itself a cover of \( E \), and so the infimum must in fact be equal to \( \mu_e(E) \).

2. Suppose that the sequential cover \( Q \) is a semi-algebra and that \( \lambda \) is a measure on \( Q \), and show that the associated outer measure is regular.

**Proof.** This is essentially the content of Proposition II.10.1 (p. 85) of our text. (Note that any \( Q \in Q \) is measurable, by Proposition 9.1, and hence so are sets in \( Q_\sigma \) and \( Q_{\sigma\delta} \).) It is true that Proposition 10.1 is only stated for sets \( E \subseteq X \) with \( \mu_e(E) < \infty \), but if \( \mu_e(E) = \infty \) there is a countable collection \( \{E_n\}_{n=1}^{\infty} \subseteq Q \) such that \( E \subseteq E' := \bigcup_{n=1}^{\infty} E_n \in Q_\sigma \). By the monotonicity of \( \mu_e \), \( \mu_e(E') = \infty \) and so \( \mu(E') = \mu_e(E') \leq \mu_e(E) + \epsilon \).

3. Let \( \mu_e \) be an outer measure on the subsets of \( X \) and let \((X, \mathcal{A}, \mu)\) be the measure space formed from \( \mu_e \) by the Caratheodory construction. Show that the outer measure \( \mu^+_e \) associated with \((X, \mathcal{A}, \mu)\) satisfies \( \mu^+_e \geq \mu_e \) (meaning that \( \mu^+_e(E) \geq \mu_e(E) \) for every \( E \subseteq X \)), and that \( \mu^+_e = \mu_e \) if and only if \( \mu_e \) is regular. Conclude that an outer measure is regular if and only if it is associated with a pair \((Q, \lambda)\), where \( Q \) is a sequential cover which is also a semi-algebra, and \( \lambda \) is a measure on \( Q \).

**Proof.** By definition,

\[
\mu^+_e(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu_e(E_n) | E \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and } E_n \in \mathcal{A} \right\}.
\]

By Question 1, on the other hand, \( \mu_e(E) \) can be computed in the same manner—the only difference being that \( \mathcal{A} \) is replaced by \( 2^X \). This means that the infimum used to compute \( \mu_e(E) \) is taken over a larger collection of covers, and so is no larger than that used to compute \( \mu^+_e(E) \). Thus, \( \mu^+_e \geq \mu_e \). (Thanks to Kamilla Kasymova for this nice argument.)

Since \( \mu^+_e \) is regular by Question 2, if \( \mu^+_e = \mu_e \) then certainly \( \mu_e \) is regular. Conversely, suppose \( \mu_e \) is regular, and let \( E \subseteq X \). Then for any \( \epsilon > 0 \) there is some \( E' \in \mathcal{A} \) with \( E \subseteq E' \) and \( \mu(E') \leq \mu_e(E) + \epsilon \). Using \( \{E'\} \) as a cover of \( E \), it follows that \( \mu^+_e(E) \leq \mu_e(E) + \epsilon \). Now let \( \epsilon \to 0 \).
4. Let $X = \{1, 2\}$ be a 2-point set. Construct an outer measure on the subsets of $X$ which is not regular.

**Proof.** If we start by assigning $\mu_e(\emptyset) = 0$, $\mu_e(\{1\}) = 1$, $\mu_e(\{1\}) = 1$ and $\mu_e(X) = c$, we need $c \geq 1$ for $\mu_e$ to be monotone and $c \leq 2$ for $\mu_e$ to be subadditive. We know that $\emptyset$ and $X$ will automatically be $\mu_e$-measurable; a check shows that $\{1\}$ (and hence also $\{2\}$) fails to be measurable if $c < 2$. In this case, the only measurable set $E'$ which covers $\{1\}$ is $E' = X$, and the inequality $\mu(E') \leq \mu_e(\{1\}) + \epsilon$ cannot be satisfied for any $\epsilon < 2 - c$.

**PART 2: CONSTRUCTION OF MEASURES**

5. Show that the saturation of a measure space is a saturated measure space; i.e. show that $\mathcal{A}_loc$ is a $\sigma$-algebra, that $\mathcal{P}$ is a measure on $\mathcal{A}_loc$, and that $(X, \mathcal{A}_loc, \mathcal{P})$ is saturated.

**Proof.** First, we show that $\mathcal{A}_loc$ is a $\sigma$-algebra:

- Clearly, $\emptyset \in \mathcal{A}_loc$.
- Let $E_1, E_2, \ldots \in \mathcal{A}_loc$. For any $A \in \mathcal{A}$ with $\mu(A) < \infty$, 
  \[(\bigcup_n E_n) \cap A = \bigcup_n (E_n \cap A) \in \mathcal{A},\]
  so $\bigcup_n E_n \in \mathcal{A}_loc$.
- Let $E \in \mathcal{A}_loc$. For any $A \in \mathcal{A}$ with $\mu(A) < \infty$, 
  \[E^c \cap A = A \setminus (E \cap A) \in \mathcal{A},\]
  so $E^c \in \mathcal{A}_loc$.

Therefore $\mathcal{A}_loc$ is a $\sigma$-algebra.

Next, we show that $\mathcal{P}$ is a measure on $\mathcal{A}_loc$. It is clear that $\mathcal{P}$ is defined and non-negative in $\mathcal{A}_loc$, and that $\mathcal{P}(A) < \infty$ for some $A \in \mathcal{A}_loc$ (for example, $\mathcal{P}(\emptyset) = 0$). It remains to show that $\mathcal{P}$ is countably additive on $\mathcal{A}_loc$; i.e. that $\mathcal{P}(\bigcup_n B_n) = \sum_n \mathcal{P}(B_n)$ for every sequence $\{B_n\}$ of pairwise disjoint sets from $\mathcal{A}_loc$. This is clear if every $B_n$ is in $\mathcal{A}$, since $\mathcal{P} = \mu$ for such sets. If even one $B_n$ is in $\mathcal{A}_loc \setminus \mathcal{A}$, then $\mathcal{P}(\bigcup_n B_n) = \infty$, and the countable additivity follows. (Since $\mathcal{A}_loc$ is a $\sigma$-algebra, we know $\mathcal{P}(\bigcup_n B_n) \in \mathcal{A}_loc$. Thus, if $\mathcal{P}(\bigcup_n B_n) < \infty$, we would have $\mathcal{P}(\bigcup_n B_n) \in \mathcal{A}$, and so each $A_n = A_n \cap (\bigcup_n B_n) \in \mathcal{A}$.)

Finally, we show that $(X, \mathcal{A}_loc, \mathcal{P})$ is saturated. To this end, suppose that $E \subseteq X$ is $\mathcal{P}$-locally measurable. This means that for every $B \in \mathcal{A}_loc$ with $\mathcal{P}(B) < \infty$, $E \cap B \in \mathcal{A}_loc$. We wish to show that $E \in \mathcal{A}_loc$, so let $A \in \mathcal{A}$ with $\mu(A) < \infty$; then $A \in \mathcal{A}_loc$ with $\mathcal{P}(A) = \mu(A) < \infty$, and so $E \cap A \in \mathcal{A}_loc$. But then $\mathcal{P}(E \cap A) \leq \mathcal{P}(A) < \infty$ implies $E \cap A \in \mathcal{A}$. It follows that $E \in \mathcal{A}_loc$ as required, and $(X, \mathcal{A}_loc, \mathcal{P})$ is saturated.

6. Show that every $\sigma$-finite measure space is saturated.

**Proof.** Let $X = \bigcup_{n=1}^\infty X_n$, where each $X_n$ is measurable, with $\mu(X_n) < \infty$. If $E \subseteq X$ is locally measurable, then each $E \cap X_n$ is measurable, and so is $E = \bigcup_n (E \cap X_n)$. 

7. Give an example of a measure space which is saturated but not complete.

Proof. The example I had in mind was Lebesgue measure restricted to Borel sets (in \( \mathbb{R}^n \), or just \( \mathbb{R} \).) This is not complete, by Proposition II.14.2 (p. 91). However, it is \( \sigma \)-finite, and hence saturated by Question 5.

Some of you came up with simpler examples. For example, let \( X = \{1, 2\} \), let \( \mathcal{A} = \{\emptyset, X\} \) and define \( \mu(\emptyset) = \mu(X) = 0 \). This space is not complete, since \{1\} is a non-measurable subset of a measure zero set. In fact, \{1\} is not even locally measurable, because \( \{1\} \cap X = \{1\} \notin \mathcal{A} \). Similarly, \{2\} is not locally measurable, and so the only locally measurable sets are \( \emptyset \) and \( X \). Since they are both measurable, \( (X, \mathcal{A}_{loc}, \mu) \) is saturated. (Thanks to Cindy Nichols for this example.)

8. Let \( X \) be an uncountable set, let \( \mathcal{A} \) consist of all subsets of \( X \) which are either countable or have countable complement, and let \( \mu \) be the function which counts the number of elements of any set \( A \in \mathcal{A} \). Show that \( (X, \mathcal{A}, \mu) \) is complete but not saturated. (To avoid tedium, you may assume that \( (X, \mathcal{A}, \mu) \) is a measure space.) What is the saturation of \( (X, \mathcal{A}, \mu) \)? Show that \( (X, \mathcal{A}, \mu) \) cannot be produced from any outer measure by the Caratheodory construction.

Proof. \( (X, \mathcal{A}, \mu) \) is complete, because the only subset of measure 0 is the empty set, all of whose subsets are measurable. To see that \( (X, \mathcal{A}, \mu) \) fails to be saturated, first note that every set is locally measurable: any set \( A \in \mathcal{A} \) with \( \mu(A) < \infty \) is a finite set, and so \( E \cap A \) is finite, and hence measurable, for any subset \( E \subseteq X \). However any set \( E \) such that neither \( E \) nor \( E^c \) is countable fails to be measurable, and so if we can find even one such set our space is not saturated. The existence of such subsets of \( X \) is intuitively obvious (?), and can be proved for any uncountable \( X \) using the Axiom of Choice. Since our principal aim here is to show that completeness and saturation are independent, we could content ourselves with exhibiting a single example. We could, for example, take \( X \) to be the interval \([0, 2] \subset \mathbb{R} \) (remember that the measure is counting measure) and take \( E = [0, 1] \).

We saw above that every subset of \( x \) is locally measurable. It follows that the saturation of \( (X, \mathcal{A}, \mu) \) is \( (X, 2^X, \overline{\mu}) \), where \( \overline{\mu}(E) = \infty \) for any \( E \notin \mathcal{A} \); i.e., \( \overline{\mu} \) is ‘true’ counting measure: the function which assigns to every subset \( E \subseteq X \) the number of elements in \( E \).

Finally, suppose that \( \mu_e \) is an outer measure which produces the measure space \( (X, \mathcal{A}, \mu) \), and let \( E \subseteq X \). If \( E \) is a finite set it is measurable, and \( \mu_e(E) = \mu(E) \) is the number of elements in \( E \). If \( E \) is an infinite set, it has finite subsets with arbitrarily many elements; i.e., for any \( n \in \mathbb{N} \) there is a subset \( E_n \subseteq E \) with at least \( n \) elements. But then \( \mu_e(E) \geq \mu_e(E_n) \geq n \) for any \( n \), and so we must have \( \mu_e(E) = \infty \). Putting the two cases together, we see that \( \mu_e \) must in fact be ‘true’ counting measure. But every subset of \( X \) is measurable with respect to counting measure, and so counting measure also fails to produce \( (X, \mathcal{A}, \mu) \).

9. Show that any measure space \( (X, \mathcal{A}, \mu) \) produced by the Caratheodory construction from a regular outer measure \( \mu_e \) must be complete and saturated.

Proof. That \( (X, \mathcal{A}, \mu) \) must be complete follows from Proposition II.6.1(iii) (p. 77). It is in fact stated explicitly in Proposition II.6.2 (p. 79).

To see that \( (X, \mathcal{A}, \mu) \) must be saturated, let \( E \) be a locally measurable set. Recall from Question 5 that the locally measurable sets from a \( \sigma \)-algebra \( \mathcal{A}_{loc} \), and so \( E^c \in \mathcal{A}_{loc} \). To
show that $E$ is measurable, it suffices to prove that for any subset $B \subseteq X$, 

$$\mu_e(B) \geq \mu_e(B \cap E) + \mu_e(B \setminus E).$$

Since this inequality is automatic if $\mu_e(B) = \infty$, we may assume $\mu_e(E) < \infty$. But then, since $\mu_e$ is regular, for any $\epsilon > 0$ there is a measurable set $B' \in \mathcal{A}$ such that $B \subseteq B'$ and $\mu(B') \leq \mu_e(B) + \epsilon$. It follows that $B' \cap E$ and $B' \setminus E = B' \cap E^c$ are both measurable, and

$$\mu_e(B \cap E) + \mu_e(B \setminus E) \leq \mu(B' \cap E) + \mu(B' \setminus E) = \mu(B') \leq \mu_e(B) + \epsilon.$$

Now let $\epsilon \to 0$.  

10. Given a measure space $(X, \mathcal{A}, \mu)$, we may consider $\mathcal{A}$ as a sequential cover of $X$ and construct the outer measure associated with $\mathcal{A}$ and $\mu$. Show that the outer measure associated in this way with $(X, \mathcal{A}, \mu)$ is the same as that associated to the completion of $(X, \mathcal{A}, \mu)$.

**Proof.** Let $(X, \mathcal{A}', \mu')$ be the completion of $(X, \mathcal{A}, \mu)$, and define outer measures $\mu_e$ and $\mu'_e$ by

$$\mu_e(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(\mathcal{E}) | \mathcal{E} \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and } E_n \in \mathcal{A} \right\}$$

(2)

and

$$\mu'_e(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu'(\mathcal{E}) | \mathcal{E} \subseteq \bigcup_{n=1}^{\infty} E_n \text{ and } E_n \in \mathcal{A}' \right\}$$

(3)

Since both $\mathcal{A}$ and $\mathcal{A}'$ are $\sigma$-algebras, the argument used in the proof of Question 1 shows that the infima in (2) and (3) are respectively equal to

$$\mu_e(E) = \inf \left\{ \mu(\mathcal{E}) | \mathcal{E} \subseteq \tilde{E} \text{ and } \tilde{E} \in \mathcal{A} \right\}$$

(4)

and

$$\mu'_e(E) = \inf \left\{ \mu'(\mathcal{E}') | \mathcal{E}' \subseteq \tilde{E}' \text{ and } \tilde{E}' \in \mathcal{A}' \right\}$$

(5)

Now, $\mathcal{A}' \supseteq \mathcal{A}$ and so the infimum in (5) is taken over a larger set than that in (4), and we certainly have $\mu'_e \leq \mu_e$. On the other hand, if $\tilde{E}' \in \mathcal{A}'$ with $E \subseteq \tilde{E}'$, then $\tilde{E}' = \tilde{E} \cup N$ for some $\tilde{E} \in \mathcal{A}$ and some $N \subseteq \tilde{N} \in \mathcal{A}$ with $\mu(\tilde{N}) = 0$; moreover, $\mu'_e(\tilde{E}') = \mu_e(\tilde{E}) = \mu_e(\tilde{E} \cup N)$. Since $E \subseteq \tilde{E} \cup N$, it follows that $\mu_e \leq \mu'_e$.  

11. Starting with $(X, \mathcal{A}, \mu)$, form the associated outer measure as in the last exercise, and then apply the Caratheodory construction. Show that the resulting measure space is the saturation of the completion of $(X, \mathcal{A}, \mu)$.

**Proof.** Note that by Question 10 the outer measure $\mu_e$, and hence the space $(X, \overline{\mathcal{A}}, \overline{\mu})$, produced from $(X, \mathcal{A}, \mu)$ is the same as that produced from the completion of $(X, \mathcal{A}, \mu)$. It is therefore sufficient to assume that $(X, \mathcal{A}, \mu)$ is complete, and to show that in this case $(X, \overline{\mathcal{A}}, \overline{\mu})$ is the saturation of $(X, \mathcal{A}, \mu)$.

Before we embark on the various stages of the proof, let us make a few preliminary observations.

(a) We know from Proposition II.9.1 (p. 83) that $\mathcal{A} \subseteq \overline{\mathcal{A}}$ and that $\overline{\mu}|_{\mathcal{A}} = \mu$.  

(b) The outer measure $\mu_e$ is constructed from $(X, \mathcal{A}, \mu)$ as in (2), which by the argument in the proof of Question 1 yields (4). This implies that for each $E \subseteq X$, and every $n \in \mathbb{N}$, there is a set $\tilde{E}_n \in \mathcal{A}$ such that $E \subseteq \tilde{E}_n$ and $\mu(\tilde{E}_n) \leq \mu_e(E) + \frac{1}{n}$. Putting $F = \bigcap_{n=1}^{\infty} \tilde{E}_n \in \mathcal{A}$, we see that for any $E \subseteq X$ there is a $\mu$-measurable set $F \in \mathcal{A}$ such that $E \subseteq F$ and $\mu(F) = \mu_e(E)$.

(c) Since $\overline{\mu}$ is produced from $\mu_e$ by the Caratheodory construction, we know that any set of $\mu_e$-outer measure 0 is $\overline{\mu}$ measurable. In fact more is true: any such set must be $\mu$-measurable. To see this, suppose that $\mu_e(E) = 0$. By the previous remark, there is a set $A \in \mathcal{A}$ with $E \subseteq A$ and $\mu(A) = 0$. Since $\mu$ is complete, this implies that $E \in \mathcal{A}$.

Turning to the proof of Question 11, the first step is to show that $\overline{\mathcal{A}} = \mathcal{A}_{loc}$, the $\sigma$-algebra of $\mu$-locally measurable sets. We will do this by showing that each of these sets is a subset of the other.

First, let $E \in \mathcal{A}_{loc}$; i.e. $E$ is $\mu$-locally measurable. We will show that $E$ is $\overline{\mu}$-locally measurable—since $\overline{\mu}$ is saturated, this will imply that $E \in \overline{\mathcal{A}}$. Suppose, therefore, that $B \in \overline{\mathcal{A}}$ with $\overline{\mu}(B) < \infty$. By Remark (b) above, there is a set $A \in \mathcal{A}$ with $B \subseteq A$ and $\mu(A) = \overline{\mu}(B) < \infty$. We can write

$$E \cap B = (E \cap A) \setminus (A \setminus B).$$

Note that $E \cap A \in \mathcal{A}$, since $E$ is $\mu$-locally measurable. Also, since $A$ and $B$ are both $\overline{\mu}$-measurable, $\mu_e(A \setminus B) = \overline{\mu}(A \setminus B) = \mu(A) - \overline{\mu}(B) = 0$, and so $A \setminus B \in \mathcal{A}$, by Remark (c). It follows that $E \cap B \in \mathcal{A} \subseteq \overline{\mathcal{A}}$, and $E$ is locally $\overline{\mu}$-measurable. This completes the proof that $\mathcal{A}_{loc} \subseteq \overline{\mathcal{A}}$.

Conversely, suppose that $E$ is $\overline{\mu}$-measurable. We will show that $E$ is $\mu$-locally measurable. Let $A \in \mathcal{A}$ with $\mu(A) < \infty$. Then $\mu_e(E \cap A) \leq \mu(A) < \infty$, and so there is a set $F \in \mathcal{A}$ with $E \cap A \subseteq F$ and $\mu(F) = \mu_e(E \cap A) < \infty$. Since $E$, $A$ and $F$ are all $\overline{\mu}$-measurable, it follows that $\mu_e(F \setminus (E \cap A)) = \overline{\mu}(F) - \overline{\mu}(E \cap A) = 0$, and so $F \setminus (E \cap A) \in \mathcal{A}$, by Remark (c). But then also $E \cap A = F \setminus (F \setminus (E \cap A)) \in \mathcal{A}$. We have shown that $\overline{\mathcal{A}} \subseteq \mathcal{A}_{loc}$, and hence that $\overline{\mathcal{A}} = \mathcal{A}_{loc}$.

We already know that $\overline{\mu}|_A = \mu$, so it only remains to show that $\overline{\mu}(E) = \infty$ for any $E \in \overline{\mathcal{A}} \setminus \mathcal{A}$. But suppose that $E \in \overline{\mathcal{A}}$ and $\overline{\mu}(E) < \infty$. Then there is a set $F \in \mathcal{A}$ such that $E \subseteq F$ and $\mu(F) = \overline{\mu}(E) < \infty$. Since $E$ and $F$ are both $\overline{\mu}$-measurable, $\mu_e(F \setminus E) = \overline{\mu}(F) - \overline{\mu}(E) = 0$, and so $F \setminus E \in \mathcal{A}$. But then $E = F \setminus (F \setminus E) \in \mathcal{A}$, and the required result follows on taking the contrapositive.

12. Show that a measure space arises from some regular outer measure by the Caratheodory construction if and only if it is complete and saturated.

**Proof.** We saw in Question 9 that any space arising by the Caratheodory construction from a regular outer measure is complete and saturated. Conversely, suppose that $(X, \mathcal{A}, \mu)$ is complete and saturated. By Question 3, the outer measure $\mu_e$ constructed from $(X, \mathcal{A}, \mu)$ by covering is regular, and by Question 11 the measure space induced by $\mu_e$ is the saturation of the completion of $(X, \mathcal{A}, \mu)$. But since $(X, \mathcal{A}, \mu)$ is complete and saturated, this space is just $(X, \mathcal{A}, \mu)$ itself.