When we are searching for a root of a polynomial, it is often convenient to start by finding bounds for any possible root. By this, we mean finding an interval which is guaranteed to contain any (real) root of the polynomial. For example, if we have somehow managed to determine that any root must lie in the interval \((-100, 100)\), we would obviously be wasting our time searching for a root near 150. Your textbook gives one method of finding such bounds, but it depends on synthetic division. This note will show you another method. We will first state the bounds as a theorem, then give some examples on how the theorem is used, then give the proof of the theorem.

**Theorem** Let \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) \((n \geq 1, a_n \neq 0)\) be a polynomial. If \( r \) is any root of \( P(x) \), then \(|r|^n\) is no larger than the largest of the numbers

\[
1, n\frac{|a_{n-1}|}{|a_n|}, n\frac{|a_{n-2}|}{|a_n|}, \ldots, n\frac{|a_1|}{|a_n|}, n\frac{|a_0|}{|a_n|}.
\]

**Example 1** Let \( P(x) = 5x^3 + 2x - 7 \). Then \( n = 3, a_3 = 5, a_2 = 0, a_1 = 2 \) and \( a_0 = -7 \). We have to compare the numbers 1 and

\[
3\frac{|a_2|}{|a_3|} = 3\frac{0}{5} = 0,
3\frac{|a_1|}{|a_3|} = 3\frac{2}{5} = \frac{6}{5},
3\frac{|a_0|}{|a_3|} = 3\frac{7}{5} = \frac{21}{5}.
\]

Taking the largest of these numbers (including 1), we see that any root \( r \) of our polynomial must satisfy \(|r| \leq 21/5\).

**Example 2** Let \( P(x) = x^4 - 2x^3 - 10x^2 + 40x - 90 \). (This example is done on page 313 of CAT, using synthetic division, and it is found that any root \( r \) must satisfy \(|r| \leq 5\).) Using our theorem, we compute the numbers

\[
1, 4\frac{2}{1} = 8, 4\frac{10}{1} = 40, 4\frac{40}{1} = 160, 4\frac{90}{1} = 360,
\]

so that any root must satisfy \(|r| \leq 360\). This is certainly not as good as the bounds obtained by the synthetic division method, but the difference is not as important as it may seem at first: once we know that any root must lie between -360 and 360, we can get down to a smaller interval very quickly with repeated bisection.

It remains to give the proof of the theorem. We start by writing

\[
P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0
\]
\[
= a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \frac{a_{n-2}}{a_n x^2} + \cdots + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)
= a_n x^n \left(1 + \frac{b_{n-1}}{x} + \frac{b_{n-2}}{x^2} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}\right),
\]

(1)

where we have written, for example, \(b_{n-1} = a_{n-1}/a_n\).

Now let \(M\) be the largest of the numbers \(1, n|b_{n-1}|, n|b_{n-2}|, \ldots, n|b_1|, n|b_0|\); note that \(M\) is the bound stated in the theorem, so we want to show that any root \(r\) of \(P(x)\) must satisfy \(|r| \leq M\).

Suppose \(|x| > M\). Then certainly \(|x| > 1\), and so \(|x^k| \geq |x|\) for every \(k = 1, 2, 3, \ldots, n - 1\). Therefore, for each such value of \(k\),

\[
\frac{|b_{n-k}|}{|x|^k} \leq \frac{|b_{n-k}|}{|x|} \leq \frac{|b_{n-k}|}{n|b_{n-k}|} = \frac{1}{n}.
\]

Observe also that the strict inequality holds, unless \(k = 1\) or \(b_{n-k} = 0\). Repeated application of the triangle inequality \(|a + b| \leq |a| + |b|\) leads to

\[
\left|\frac{b_{n-1}}{x} + \frac{b_{n-2}}{x^2} + \ldots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n}\right| \leq \left|\frac{b_{n-1}}{x}\right| + \left|\frac{b_{n-2}}{x^2}\right| + \cdots + \left|\frac{b_1}{x^{n-1}}\right| + \left|\frac{b_0}{x^n}\right| \leq n \cdot \frac{1}{n} = 1,
\]

with strict inequality holding unless every \(b_{n-k} = 0\). In this special case, however, it is easy to see that the theorem is true (exercise). In particular, we see that

\[
\frac{b_{n-1}}{x} + \frac{b_{n-2}}{x^2} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n} \neq -1,
\]

so that

\[
1 + \frac{b_{n-1}}{x} + \frac{b_{n-2}}{x^2} + \cdots + \frac{b_1}{x^{n-1}} + \frac{b_0}{x^n} \neq 0.
\]

It now follows from (1), since \(a_n \neq 0\) and \(|x| \geq 1\), that \(P(x) \neq 0\).

We have shown that if \(|x| > M\), then \(P(x) \neq 0\); this is equivalent to the statement of the theorem.