Accelerating SENSE Using Compressed Sensing

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Abstract

Both parallel magnetic resonance imaging (pMRI) and compressed sensing (CS) are emerging techniques to accelerate conventional MRI by reducing the number of acquired data. The combination of pMRI and CS for further acceleration is of great interests. In this paper, we propose two methods to combine SENSE, one of the standard methods for pMRI, and SparseMRI, a recently proposed method for CS-MRI with Cartesian trajectories. The first method, named SparseSENSE, directly formulates the reconstruction from multi-channel reduced $k$-space data as the same nonlinear convex optimization problem as SparseMRI, except that the encoding matrix is the Fourier transform of the channel-specific sensitivity modulation. The second method, named CS-SENSE, first employs SparseMRI to reconstruct a set of aliased reduced-field-of-view images in each channel, and then applies Cartesian SENSE to reconstruct the final image. The results from simulations, phantom and in vivo experiments demonstrate that both SparseSENSE and CS-SENSE can achieve a reduction factor higher than those achieved by SparseMRI and SENSE individually, and CS-SENSE outperforms SparseSENSE in most cases.

Key words: SENSE, Magnetic resonance imaging, Compressed Sensing, SparseSENSE, CS-SENSE
INTRODUCTION

MR imaging speed is usually limited by the large number of samples needed along the phase encoding direction. In conventional MRI using Fourier encoding, the required number of samples is determined by the field of view and the resolution of the image based on the Shannon sampling theory. To accelerate the conventional MRI, both parallel magnetic resonance imaging (pMRI) and compressed sensing MRI (CS-MRI) are advanced techniques to reduce the number of acquired data. Different from the methods that rely on dynamic scans to reduce acquisition, such as Keyhole (1, 2), RIGR (3), DIME (4), UNFOLD (5), and k-t Blast (6), pMRI and CS-MRI are based on advanced sampling theory instead of the Shannon sampling theory such that no dynamic scans are needed.

PMRI is based on Papoulis generalized sampling theory (7). In pMRI, due to the availability of multi-channel coils, the k-space can be sampled below the Nyquist sampling rate to reconstruct the image from these multi-channel k-space data. Standard reconstruction methods include SENSE (8), SMASH (9), GRAPPA (10), etc. Theoretically, the maximum reduction factor can be equal to the number of channels under ideal conditions. However, this maximum usually cannot be achieved due to practical limitations such as noise and imperfect coil geometry.

CS-MRI is based on CS theory (11-15), a new framework for data sampling and signal recovery. CS-MRI takes advantage of the fact that MRI meets the two conditions of CS: the MR images are sparse after certain transformations and the Fourier encoding is incoherent with some sparse transformations. Therefore, the MR images can be reconstructed using a nonlinear convex program from data sampled at a rate close to their intrinsic information rate which is well below the Nyquist rate. CS-MRI methods include SparseMRI for Cartesian trajectories (16) and others for radial trajectories (17, 18).

Because pMRI and CS-MRI reduce sampling based on different ancillary information (channel sensitivities for pMRI and image sparseness for CS), it is desirable to combine pMRI and CS for further reduction. In this paper, we systematically study this topic and propose two methods for the Cartesian case. The straightforward method, named SparseSENSE, reconstructs image from the multi-channel data using the same nonlinear convex program as that of SparseMRI, except that the Fourier encoding is replaced by the sensitivity encoding. SparseSENSE has been partially presented in Ref (19) and similar ideas have also been independently studied in Refs. (20-22). The alternative method, named CS-SENSE, sequentially carries out SparseMRI for reconstructing the aliased image in each channel and then SENSE for the final image. CS-SENSE has been partially presented in Ref (23), and is different from the method in Ref (24) which uses CS to reconstruct the full field of view images from each channel and combined them to form the final image. We analyze the SparseSENSE and CS-SENSE theoretically and compare them numerically using both simulation and experiments. Our results show that both SparseSENSE and CS-SENSE can achieve a reduction factor higher than those achieved by SparseMRI and SENSE.
individually, and CS-SENSE may outperform SparseSENSE in most cases.

**THEORY**

*Summary of SENSE*

SENSE is one of the standard reconstruction methods for parallel imaging. For arbitrary trajectories, the general SENSE equation is

\[
Ef = d
\]  

[1]

where \(d\) is the vector formed from \(k\)-space data acquired in all channels, \(f\) is the unknown vector defining the desired full field-of-view (FOV) image to be computed, both with a lexicographical row ordering of the two-dimensional array components, and \(E\) is the sensitivity encoding matrix whose entries are

\[
E_{i(m,n)} = e^{-i2\pi(k_xk_x, k_yk_y)}s_l(x, y)
\]  

[2]

where \(k_x\) and \(k_y\) denote the \(k\)-space sampling position for the \(m\)-th element in \(d\), \((x, y)\) denotes the pixel position for the \(n\)-th element in \(f\), and \(s_l\) is the sensitivity profile of the \(l\)-th receiver channel.

*Summary of SparseMRI*

CS is a new mathematical framework for signal sampling and recovery. It allows faithful recovery of a signal from measurements that are far fewer than those required by the Nyquist sampling rate. Therefore, it is of great interest to apply CS to MRI for fast imaging (16-18). To use the CS theory, two conditions need to be satisfied: (a) the image is sparse in a known transform domain and (b) the encoding waveform is incoherent with the sparsifying basis. Fortunately, conventional Fourier imaging meets the above two conditions: most MR images have a sparse representation in some transform domain (e.g., image domain or wavelet domain) and the Fourier encoding is incoherent with some sparsifying basis such as the canonical basis or the fine scales of a wavelet transform (13). Therefore, with high probability the MR images can be recovered from the randomly undersampled \(k\)-space data by solving a nonlinear convex optimization problem.

SparseMRI is a practical technique to apply CS to conventional Cartesian MRI. Considering the practical limitations, the method fully samples the readout and randomly undersamples the phase-encoding lines using a variable-density sampling scheme with denser sampling near the center of the \(k\)-space (16, 25). The randomly generated sampling pattern with the lowest level of incoherence is chosen for data acquisition. The final image is reconstructed from undersampled \(k\)-space data by solving a constrained convex optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \| \Psi f \|_h \\ 
\text{s.t.} & \quad \| f^\omega - d^\omega \|_2 \leq \varepsilon
\end{align*}
\]  

[3]
where $F^u$ and $d^u$ are the random subset of the rows of the Fourier encoding matrix and the corresponding vector formed from the undersampled $k$-space data, respectively, and $\Psi$ is the sparse transformation matrix. Parameter $\varepsilon$ controls the fidelity of the reconstruction to the measured data and usually is set below the expected noise level. This problem can be transformed to an unconstrained regularization problem

$$\arg\min_f \{ \|F^u f - d^u\|_2^2 + \lambda \|\Psi f\|_1 \}$$

where $\lambda$ is the regularization parameter which manages the tradeoff between data consistency and sparsity prior and can be selected appropriately such that the solution of Eq. [4] is exactly the same as that of Eq. [3] (16).

**Proposed SparseSENSE**

The first method, named sparseSENSE, is a direct generalization of SparseMRI in parallel imaging. In data acquisition, the same random undersampling scheme as that of SparseMRI is used to sample the $k$-space for all channels. Specifically, the $k$-space data are randomly undersampled along the phase-encoding direction with denser sampling near the center of the $k$-space. The peak interference of transform point spread function (TPSF) defined in (16, 25) is used to measure the goodness of a specific undersampling pattern. To choose a good sampling pattern, we randomly generate 20 sampling patterns and the one with the lowest peak interference is selected for data acquisition. In reconstruction, the problem is formulated as the same constrained nonlinear convex program, except that the Fourier encoding is replaced by the sensitivity encoding $E$ based on the SENSE imaging equation [1]:

$$\text{minimize } \|\Psi f\|_1 \quad \text{s.t. } \|E f - d\|_2 \leq \varepsilon$$

[5]

In solving Eq. [5], we also instead solve the equivalent unconstrained regularization problem

$$\arg\min_f \{ \|d - Ef\|_2^2 + \lambda \|\Psi f\|_1 \}$$

[6]

Please note that the incoherence between the encoding matrix $E$ and the sparsifying basis $\Psi$ is not explored here because $E$ is not an orthonormal basis primarily considered in the CS theory and is thereby beyond the scope of this paper.

**Proposed CS-SENSE**

The second approach, named CS-SENSE, decouples the CS and pMRI procedure in reconstruction and applies SparseMRI and SENSE sequentially. When the sampling position is on a uniform Cartesian grid, the general SENSE matrix in Eq. [1] can be decomposed to the product of two matrices as

$$E = \tilde{F}\tilde{C}$$

[7]
where

\[
\tilde{\mathbf{F}} = \begin{bmatrix}
  \mathbf{F} & 0 & \cdots & 0 \\
  0 & \mathbf{F} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \mathbf{F}
\end{bmatrix}
\]

with each block \( \mathbf{F} \) being the 2D Fourier matrix of the reduced FOV and the number of such blocks equals to the number of channels \( L \), and \( \tilde{\mathbf{C}} \) is a sensitivity modulation matrix

\[
\tilde{\mathbf{C}} = \begin{bmatrix}
  \mathbf{C}_{1,1} & \mathbf{C}_{1,2} & \cdots & \mathbf{C}_{1,R_2} \\
  \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \cdots & \mathbf{C}_{2,R_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  \mathbf{C}_{L,1} & \mathbf{C}_{L,2} & \cdots & \mathbf{C}_{L,R_2}
\end{bmatrix}
\]

with

\[
\mathbf{C}_{l,r} = \begin{bmatrix}
  s_l(1 + (r-1)\hat{B}_x,1) & 0 & \cdots & \cdots & \cdots & 0 \\
  0 & \ddots & 0 & \vdots & \vdots & \vdots \\
  \vdots & 0 & s_l(1 + (r-1)\hat{B}_x,B_y) & 0 & \ddots & \vdots \\
  \vdots & \ddots & 0 & \ddots & 0 & \vdots \\
  \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\
  0 & 0 & \cdots & \cdots & \cdots & s_l(r\hat{B}_x,B_y)
\end{bmatrix}
\]

and \( R_2 \) being the reduction factor, \( B_x \) and \( B_y \) being the original FOV, and \( \hat{B}_x \) \((= B_x / R_2)\) being the reduced FOV due to \( k \)-space undersampling by a factor of \( R_2 \). The matrix \( \mathbf{C}_{l,r} \) is of \( \hat{B}_x \times \hat{B}_y \) whose diagonal elements are \( s_l(x + (r-1)\hat{B}_x, y) \) with a lexicographical row ordering of the two-dimensional array \( 0 < x \leq \hat{B}_x, \ 0 < y \leq \hat{B}_y \). Therefore, Eq. [1] is decoupled to the following two equations:

\[
\tilde{\mathbf{C}} \mathbf{f} = \mathbf{A} \quad [8]
\]

where the original full FOV image, weighted by different sensitivities from all channels, is folded to generate a set of aliased images with reduced FOV, with \( \mathbf{f}^A = [\mathbf{f}_1^A, \mathbf{f}_2^A, \ldots, \mathbf{f}_L^A]^T \) and \( \mathbf{f}_l^A \) being a vector for the aliased reduced FOV images from the \( l \)-th channel, and

\[
\tilde{\mathbf{F}} \mathbf{f}^A = \mathbf{d} \quad [9]
\]

where the Fourier encoding operation applies Fourier transform on the aliased images \( \mathbf{f}^A \) with reduced FOV.

When channel-wise decomposed, Eq. [9] becomes
for the $l$-th channel, where $\mathbf{d}_l$ is the acquired reduced $k$-space data from the $l$-th channel. Similarly, Eq. [8] can also be pixel-wise decomposed to a number of small equations for $0 < x \leq \hat{B}_x$, $0 < y \leq B_y$:

$$\tilde{\mathbf{C}}_{(x,y)} \mathbf{f}_{(x,y)} = \mathbf{f}_A$$

where

$$\mathbf{f}_{(x,y)} = \begin{bmatrix}
  f(x,y) \\
  f(x + \hat{B}_x, y) \\
  \vdots \\
  f(x + (R_x - 1)\hat{B}_x, y)
\end{bmatrix}, \quad \mathbf{f}_A = \begin{bmatrix}
  f_1^A(x,y) \\
  f_2^A(x,y) \\
  \vdots \\
  f_{R_x}^A(x,y)
\end{bmatrix},$$

and $f_i^A(x,y)$ is the value of the aliased images $\mathbf{f}_A$ with reduced FOV of $\hat{B}_x$ at pixel $(x,y)$ from the $l$-th channel. This decoupled formulation is exactly what the image-domain basic SENSE reconstruction is based on: first obtain a set of aliased images of all channels by Fourier transform according to Eq. [10] and then reconstruct the desired image by solving Eq. [11].

The proposed CS-SENSE method also takes advantage of the decoupled formulation and uses a similar procedure except that SparseMRI is applied to the first step due to the Fourier encoding matrix in Eq. [10]. Specifically, the same random sampling scheme as that for SparseMRI is employed to further undersample the phase encoding lines that are already reduced for the aliased images with reduced FOV. One out of 20 randomly generated sampling patterns with the lowest peak TPSF is chosen to acquire the data. With this random undersampling, Eq. [10] is represented as

$$\mathbf{F}^u f_i^A = \mathbf{d}_i^u$$

where $\mathbf{d}_i^u$ is the undersampled $k$-space data from the $i$-th channel and is a subset of $\mathbf{d}_i$. Therefore, the aliased image $f_i^A$ at each channel can be reconstructed by solving

$$\begin{aligned}
\text{minimize} & \|\Psi f_i^A\|_1 \quad \text{s.t.} \quad \|\mathbf{F}^u f_i^A - \mathbf{d}_i^u\|_2 \leq \varepsilon \\
\text{which is equivalent to} & \quad \arg \min_{f_i^A} \left\{ \left\| \mathbf{F}^u f_i^A - \mathbf{d}_i^u \right\|_2^2 + \lambda \|\Psi f_i^A\|_1 \right\}.
\end{aligned}$$

[10]

[11]

[12]
With the aliased images from all channels, the desired full FOV image $f$ can be reconstructed pixel by pixel using the image domain basic SENSE method, which finds the least squares solution to Eq. [11]. It is easy to see that the net reduction factor $R$ of CS-SENSE is equal to the product of the reduction factor $R_1$ in SparseMRI and the reduction factor $R_2$ in Cartesian SENSE, i.e., $R = R_1 \times R_2$.

**Comparison of the Two Proposed Methods**

Both the sampling scheme and the reconstruction algorithm of the two proposed methods are different. SparseSENSE randomly undersamples the full $k$-space, while CS-SENSE randomly undersamples the phase encoding lines which is already reduced for the aliased images. SparseSENSE directly solves for the desired full FOV image using a nonlinear program, while CS-SENSE sequentially solves for the aliased reduced FOV images using a nonlinear program and the final desired full FOV image using a least square solution. Both methods have their own strengths and weakness.

First, the SparseSENSE method is more straightforward and less complex than the CS-SENSE method. The former only changes the encoding matrix of SparseMRI from the Fourier matrix to the sensitivity encoding matrix and keeps the sampling scheme and the rest of the reconstruction formula unchanged. While the latter carries out a number of SparseMRI independently for all channels and thus requires more computations.

Second, SparseSENSE can be used for non-Cartesian trajectories because it is based on the general SENSE equation, while the current formulation of CS-SENSE can only be used for Cartesian trajectories.

Third, the full FOV image that SparseSENSE reconstructs is usually sparser in the transform domain than the aliased reduced FOV image that CS-SENSE reconstructs. Here the sparsity level is defined as the percentage of nonzero elements in the entire signal as in Ref (26). Figure 1 gives an example to compare the sparsity levels of a phantom image and its aliased image with a reduction factor of $R_2 = 2$, in two different sparse transform domains: finite-difference and wavelet. It is seen that the sparsity level of the aliased image is about $1 / R_2$ of that of the original image.

Finally and most importantly, the incoherence condition required in CS is guaranteed in CS-SENSE, but not necessarily in SparseSENSE. CS-SENSE uses the same Fourier encoding matrix as SparseMRI does, whose incoherence has been proved (11-13). In contrast, the encoding matrix in SparseSENSE is channel sensitivity dependent and can vary from scan to scan. The incoherence between the sensitivity encoding and the sparse transformation basis such as wavelet is not necessarily satisfied and is difficult to verify for each scan. In addition, CS is a method for solving underdetermined system. However, the SparseSENSE equation is usually made to be overdetermined (i.e., the reduction factor is lower than the channel number) due to the ill-conditioning problem, and thus does not fit in the CS framework. The unconstrained minimization in Eq. [6] is better regarded as regularization with a Gibbs prior, and the
improvement of SparseSENSE over SENSE is attributed to regularization. In comparison, CS-SENSE solves an underdetermined problem and therefore fits in the CS framework.

METHODS

The two proposed methods were evaluated on four datasets: simulated data without and with additive noise, scanned phantom data and scanned human brain data. All reconstruction methods were implemented in MATLAB (MathWorks, Natick, MA), and non-linear conjugate gradient (16) was used to solve Eq. [6] and [14]. To quantitatively evaluate the performance of the proposed methods, the normalized mean square error (NMSE) between the reconstructed and the reference images (27) was calculated. All reconstructed images are normalized and shown on the same scale.

Simulation without Noise

The objective of this simulation is to compare the two proposed methods at different reduction factors when the image is perfectly sparse in the transform domain and there is no measurement noise. In this simulation, a 256 × 256 numerical phantom (28) shown in Fig. 1a was used as the original image. The phantom is piecewise smooth, so that finite-difference was used as the sparse transformation. The sensitivities of an eight-channel coil were simulated using the Biot-Savart law (29). The k-space data were generated by Fourier transforming the sensitivity-weighted images and undersampled according to the designed sampling pattern. The reduction factors take \( R = 4, 6, 8 \) and 12 for SparseSENSE, and take \( R = 2 \times 2, 3 \times 2, 4 \times 2 \) and 6 × 2 for CS-SENSE, where the first factor \( R_1 \) is the reduction factor of SparseMRI and \( R_2 \) is the reduction factor of SENSE.

Simulation with Noise

To investigate the effect of noise on the two proposed methods, different levels of Gaussian noise were added to the above simulated k-space data, yielding signal-to-noise (SNR) ratios of 30, 15, and 7.5. Other conditions keep the same as the noise free case. NMSEs were calculated as functions of the reduction factor and SNR for both methods. Furthermore, the reconstructed images were also compared visually for SNR=15.

Phantom Experiment

The objective of this phantom experiment is to compare the two proposed methods in practical conditions where noise and inaccurate sensitivities can both contribute to reconstruction error. The phantom itself is sparse with a few well-defined structures, so that any artifacts and noise are clearly visible. Both the identity and the finite-difference matrices were used as the sparse transformation matrix. A T1-weighted phantom scan was performed using a 2D spin echo sequence on a 3T commercial scanner (GE Healthcare,
Waukesha, WI) with an 8-channel torso channel (TE/TR = 11/300 ms, 18cm FOV, 8 slices, 256 × 256 matrix). The full k-space data were acquired and the sum-of-square (SoS) reconstruction from all channels was used as the reference for comparison. The central 32 fully sampled phase encodings were weighted by a cosine taper window (30) and used to generate a set of low resolution images. These images were used as the channel sensitivity profiles after normalization by their SoS reconstruction. The same reduction factors as the computer simulation were used. In addition to the two proposed methods, two existing approaches were also used to reconstruct the desired image. One approach used the conjugate gradient SENSE, and the other employed SparseMRI for a full FOV image in each channel and then combined all images by SoS. Both were based on the data sampled with the same pattern as SparseSENSE. The reconstructed images were shown for comparison and the corresponding “comb” regions were zoomed to reveal details.

In vivo Human Brain Imaging Experiment
In the human brain experiment, a set of in vivo brain data was acquired on a 3T commercial scanner (GE Healthcare, Waukesha, WI) with an 8-channel head channel (Invivo, Gainesville, FL) using a 2D T1-weighted spin echo protocol (axial plane, TE/TR = 11/700 ms, 22cm FOV, 10 slices, 256 × 256 matrix). Informed consent was obtained from the volunteer in accordance with the institutional review board policy. This study compares the performance of the proposed methods for an in vivo image that is compressible (i.e., sparse after thresholding the transform coefficients) but not perfectly transform sparse. Both the Daubechie-4 wavelet and finite-difference were used as the sparse transformation.

Similar to the phantom experiment, the SoS reconstruction was used as the reference for comparison, and the central 32 fully sampled phase encodings were used to estimate the channel sensitivity profiles. The same reduction factors as before were used in SparseSENSE. In CS-SENSE, a number of different combinations were investigated for the reduction factors, including $R = 4 \ (2 \times 2)$, $R = 6 \ (2 \times 3, \ 3 \times 2)$, $R = 8 \ (2 \times 4, \ 4 \times 2)$ and $R = 12 \ (6 \times 2, \ 4 \times 3, \ 3 \times 4)$. Both the reconstructed images and their corresponding NMSE were compared.

RESULTS
All images are labeled by the method used on the top-left corner and the reduction factor on the top-right corner. “Ref” denotes the reference image, “A” the CS-SENSE method, “B” the SparseSENSE method, “C” the SparseMRI followed by SoS, and “D” the conjugate gradient SENSE.

Simulation without Noise
The results for the numerical phantom without noise are shown in Fig. 2. At $R = 4$, both methods are able to reconstruct the original image exactly. As the net reduction factor increases, both the CS-SENSE and
SparseSENSE reconstructions have more and more artifacts and become poor when the reduction factor is larger than the number of channels. CS-SENSE is seen to be slightly better than SparseSENSE in terms of artifacts.

**Simulation with Noise**

The NMSE provides a combined metric for both image noise and artifacts. Table 1 shows the NMSEs of both methods with different reduction factors and SNRs, \( R = 2 \times 2, 3 \times 2, 4 \times 2 \) and \( 6 \times 2 \) and SNR= 7.5, 15 and 30. The unit of the NMSEs is \( \times 10^{-02} \). As expected, both methods perform better as the SNR increases and/or the reduction factor decreases. CS-SENSE is superior to SparseSENSE in terms of NMSE with the same reduction factor and SNR combinations.

Figure 3 shows the reconstructed images of CS-SENSE and SparseSENSE for SNR=15. The CS-SENSE reconstructions are seen to have similar aliasing artifacts to the noise free ones, except that some random noise is added in the reconstruction. In contrast, the data noise introduces additional artifacts in SparseSENSE reconstruction, which becomes serious with large reduction factors.

**Phantom Experiment**

Figure 4 shows the reconstructions from the scanned phantom data, with the “comb” region zoomed in to reveal more details on the bottom-left corner of each image. For a moderate reduction factor (\( R = 4 \)), both proposed methods are able to reconstruct an image that is visually almost the same as the reference SoS image. (The SoS image is not shown in Fig. 4 because this similarity.) In contrast, SparseMRI followed by SoS has visible artifacts and SENSE has visibly more noise. As the reduction factor becomes larger (\( R = 6 \)), the CS-SENSE reconstruction is seen to be less blurry with more details than the SparseSENSE reconstruction, which is better shown in the zoomed “comb” region. For a reduction factor equal to the number of channels (\( R = 8 \)), most details are lost in the SparseSENSE reconstruction, but preserved in the CS-SENSE reconstruction. At an even larger reduction factor (\( R = 12 \)), both methods fail to reconstruct the phantom image faithfully.

**In vivo Human Brain Imaging Experiment**

Figure 5 shows the reconstructions of the two proposed methods when the net reduction factors are \( R = 4, 6 \) and 8. For the CS-SENSE method, we show two different combinations of \( R_1 \) and \( R_2 \) for reduction factors of \( R = 6 \) and 8. Figure 6 shows the reconstructions with a net reduction factor of \( R = 12 \), where combinations of \( R = 6 \times 2, 3 \times 4 \) and \( 4 \times 3 \) are employed for CS-SENSE.

At a reduction factor of \( R = 4 \), both reconstructions are almost the same as the reference image, except that the CS-SENSE image is slightly noisier and the SparseSENSE image is slightly more blurry.
As the reduction factor becomes larger ($R = 6$), the SparseSENSE reconstruction has more aliasing artifacts, and is more blurry with less details than the CS-SENSE reconstruction. For a high reduction factor equal to the number of channels ($R = 8$), the SparseSENSE has serious aliasing artifacts with most details lost. In comparison, the CS-SENSE method either preserves much more details with fewer aliasing artifacts but is much noisier with $R_s=4$, or preserves slightly more details with fewer aliasing artifacts but is slightly noisier with $R_s=2$. When the reduction factor is larger than the number of channels ($R=12$), none of the reconstructions are acceptable, although they have different characteristics.

For this brain data set, Tab. 2 shows the NMSEs of both methods with different combinations of reduction factors. The CS-SENSE method has less NMSE than the SparseSENSE method with the same reduction factor.

**DISCUSSION**

In this paper, two methods, SparseSENSE and CS-SENSE, are proposed to further accelerate parallel imaging using CS. The methods have different sampling schemes and reconstruction algorithms. The experimental results demonstrate both proposed methods are able to accelerate the conventional SENSE, but CS-SENSE is usually superior to SparseSENSE.

**Reconstruction Properties**

As mentioned in Theory, SparseSENSE is a regularized SENSE reconstruction when the reduction factor is not larger than the number of channels. The SENSE reconstruction depends on the condition of the overdetermined SENSE matrix (i.e., the g-factor ($8$)). If the condition of the matrix is moderately ill (e.g., when $R = 4$), the $\ell_1$ minimization in SparseSENSE can regularize the condition and reconstruct the image almost perfectly. However, the condition is usually seriously ill when the reduction factor is high (e.g., $R = 6$ and $R = 8$), and the $\ell_1$ minimization regularizes the condition but at the cost of increased artifacts and blurriness. The sparser the image is in the transform domain, the more accurate the regularization prior is and thus the better the reconstruction is. When the reduction factor is higher than the number of channels, the SENSE equation becomes underdetermined and the image reconstruction problem fits into the CS framework. Based on the CS theory, it is important to choose a transform that gives the image a sparsest representation and a sampling pattern that makes the sensitivity encoding matrix to be the most incoherent with the chosen sparsifying transform. In addition, when noise is present, the reconstruction error increases with the noise level. When the image is compressible but not perfectly transform sparse, the reconstruction error increases with the compression error defined as the difference between the original image and the compressed image with all small coefficients set to zero.
In CS-SENSE, the first step of SparseMRI and the second step of the basic SENSE are performed sequentially. So the reconstruction error of CS-SENSE comes from both SparseMRI and the basic SENSE. As mentioned in Ref (16), the achievable reduction factor depends on both the image sparsity and the incoherence of the sampling patterns under the ideal condition. This is demonstrated in the noise-free simulation where the aliased images can be perfectly reconstructed with a reduction factor of $R_1 = 2$. When the reduction factor is further increased in SparseMRI, the reconstruction has some artifacts. Under the practical condition, the reconstruction error also increases with the noise level and compression error. Any reconstruction error from SparseMRI is fed into the basic SENSE reconstruction and propagates to the final reconstruction. When the basic SENSE matrix is ill-conditioned, both the error due to noise and the error due to artifacts are amplified and the error propagation becomes serious, which is demonstrated in Fig. 7 and Fig. 8. In Fig. 7 where there is no noise, the reconstruction error is only due to artifacts propagation. Although the aliased images with both $R_1 = 2$ and $R_1 = 4$ are satisfying visually, the artifacts are amplified by SENSE more with $R_2 = 4$ than with $R_2 = 2$, because the condition of SENSE is usually deteriorated with the increased reduction factor. Similarly in Fig. 8, both noise and artifacts in the aliased images propagate to the final reconstruction through SENSE, and SENSE with $R_2 = 4$ amplifies the noise and artifacts more than SENSE with $R_2 = 2$ does. In conclusion, the reduction factors in SparseMRI and SENSE should be well balanced to minimize the final reconstruction error. When the image is sparse, a large factor of $R_1$ is preferred; when the SENSE is well-conditioned, a large factor of $R_2$ is preferred. Future work will investigate optimal combination of $R_1$ and $R_2$. To address the noise propagation due to ill-conditioned SENSE, regularization techniques suitable for Cartesian SENSE (31-33) may also be incorporated into the SENSE step of CS-SENSE.

**Computational Complexity**

The current implementation of CS-SENSE needs longer execution time than SparseSENSE due to more computations. This drawback can be overcome by parallel computing using multiprocessor or dedicated hardware systems. Because the $k$-space data are acquired from multiple channels simultaneously in pMRI and reconstruction of the aliased images at each channel is independent of each other, the SparseMRI procedure in CS-SENSE can be performed simultaneously for all channels. Thus the computational time of CS-SENSE can be reduced to be approximately the same as that of SparseSENSE. The computation time of SENSE is negligible compared to the iterative algorithms used for SparseMRI (34) and commodity graphics hardware can be utilized to compute Cartesian SENSE even faster (35).

**Extensions**
In this work, $\ell_1$ minimization is used in both SparseSENSE and CS-SENSE reconstruction. $\ell_p$ minimization ($0 < p < 1$) and approximated $\ell_0$ minimization have recently been proposed for CS reconstruction and shown to require less number of measurement data (36-38). These ideas should be directly applicable to the two proposed methods for possibly higher acceleration.

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REFERENCES

Fig. 1. The magnitude images for the transform coefficients of the original and the aliased sensitivity-weighted phantom image with different transforms, where (a) shows the original phantom and (b)-(e) show the transform coefficients. Among them, (b) and (d) are the finite-difference transform coefficients, (c) and (e) are the wavelet transform coefficients, (b) and (c) are for the original phantom, and (d) and (e) are for the aliased sensitivity-weighted phantom image with $R_2 = 2$. It can be seen that the sparsity level of aliased sensitivity weighted image is about $1 / R_2$ of that of the original image in the transform domain.
Fig. 2. Images reconstructed from a set of simulated, eight-channel, noise-free data with different net reduction factors. The left column is for CS-SENSE (denoted as “A”) with $R = 2 \times 2, 3 \times 2, 4 \times 2$ and $6 \times 2$ from top to bottom. The right column is for SparseSENSE (denoted as “B”) with $R = 4, 6, 8$ and $12$. The method and the reduction factor are shown on the top left and right corner of each image. Both CS-SENSE and SparseSENSE can reconstruct the original reference image almost exactly with a moderate reduction factor. The reconstructions of both methods present more artifacts as the reduction factor increases.
Fig. 3. Images reconstructed from a set of simulated, eight-channel, noisy data with different net reduction factors at SNR = 15dB. Similarly, the left column is for CS-SENSE (A) and the right column for SparseSENSE (B) with the reduction factor shown on the top right corner of the image. The presence of noise in measurement only brings in more noise to the noise-free CS-SENSE reconstructions in Fig. 2, but causes additional artifacts in the SparseSENSE reconstructions.
Fig. 4. Phantom images reconstructed using CS-SENSE (A), SparseSENSE (B), SparseMRI followed by SoS (C), and SENSE (D) from a set of eight-channel scanned data with different net reduction factors. The corresponding “comb” region was zoomed to reveal details. Both CS-SENSE and SparseSENSE are superior to SparseMRI and SENSE for all reduction factors. The CS-SENSE reconstruction has less aliasing artifacts, is less blurry, and preserves more details compared to the SparseSENSE reconstruction with the same reduction factor.
Fig. 5. Brain images reconstructed using CS-SENSE (A) and SparseSENSE (B) from a set of eight-channel scanned data with different reduction factors. A region of interest (enclosed by a rectangle in the reference image denoted as “Ref”) was zoomed and shown at the bottom-right corner of each image. By adjusting $R_1$ and $R_2$ combinations properly, CS-SENSE can achieve less noise, aliasing artifacts, and blurriness than SparseSENSE.
Fig. 6. Brain images reconstructed from the same set of data as the one in Fig. 5 with different combinations of reduction factors for CS-SENSE at $R = 12$. 
Fig. 7. The aliased images (left column) reconstructed from the same noise-free data in Fig. 2 using SparseMRI and the corresponding final images (right column) with different combinations of $R = 8$. When the basic SENSE matrix is ill-conditioned ($R_2 = 4$), the artifacts that are almost invisible in the aliased image are amplified and becomes visible in the final reconstruction.
Fig. 8. The same results as Fig. 7 except that the scanned phantom data set was used. Both noise and artifacts are propagated to the final reconstruction.
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